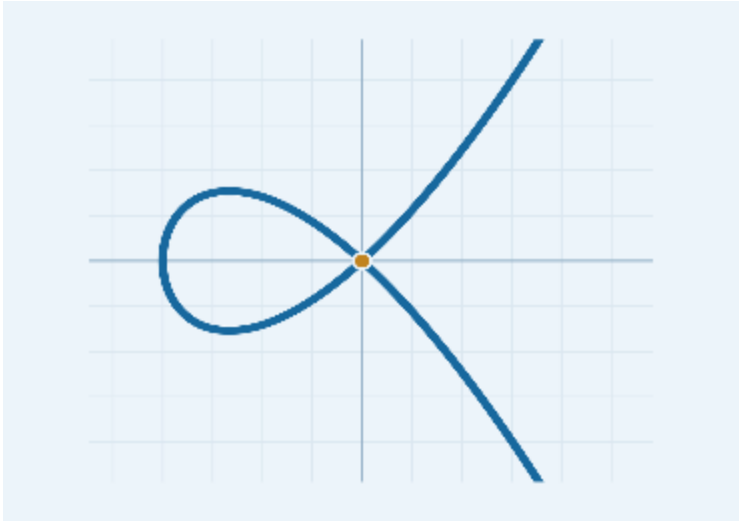
Birational Geometry and the impact of AI

UCLA College | Physical Sciences Mathematics

Joaquin Moraga
Associate Professor
UCLA Mathematics

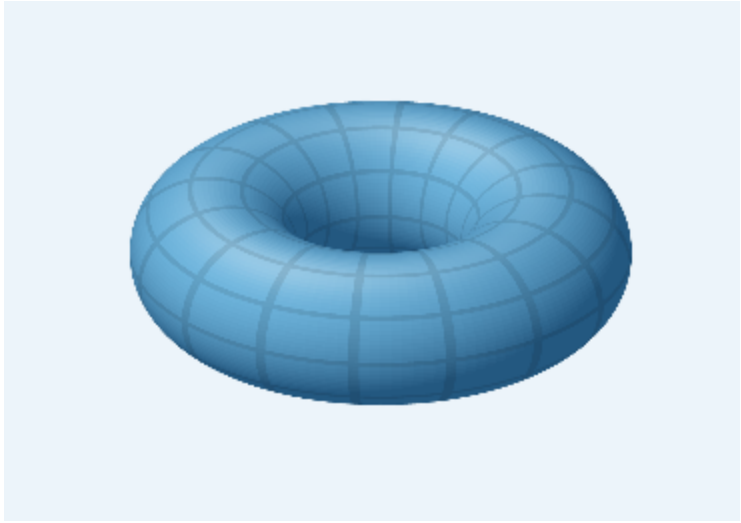
What is algebraic geometry?

Algebraic geometry is the study of geometric shapes described by one or more polynomial equations.



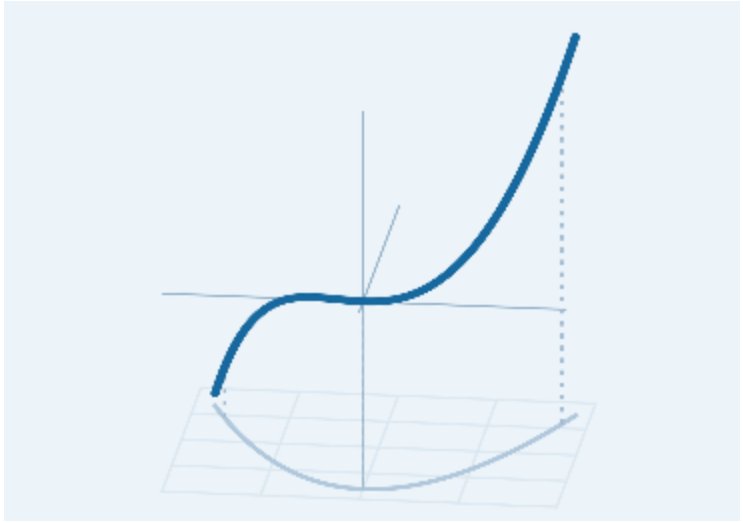
Nodal cubic

$$y^2 = x^2(x+1)$$



Torus

$$(x^2+y^2+z^2+3)^2 = 16(x^2+y^2)$$



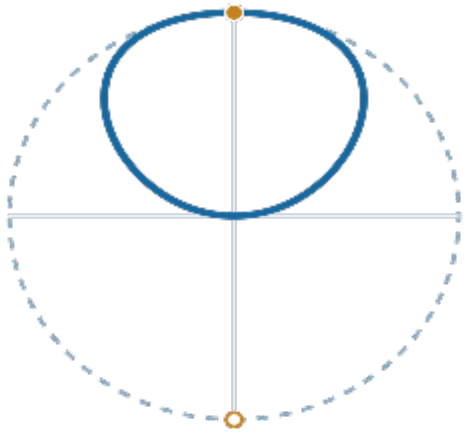
Twisted cubic

$$y = x^2, \qquad z = xy$$

Projective geometry

To understand behavior at infinity, we use projective coordinates.

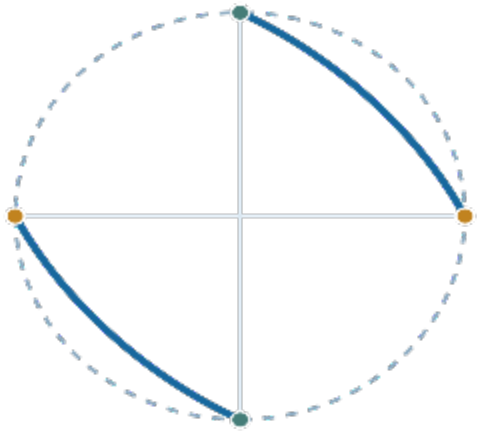
$$\mathbb{P}^n_{\mathbb{C}} = \left(\mathbb{C}^{n+1} \setminus \{(0,\dots,0)\}\right)/\mathbb{C}^* \qquad \text{with coordinates } [x_0 : \cdots : x_n]$$



Projective closure of a parabola

$$x_0 x_2 = x_1^2$$

One point at infinity



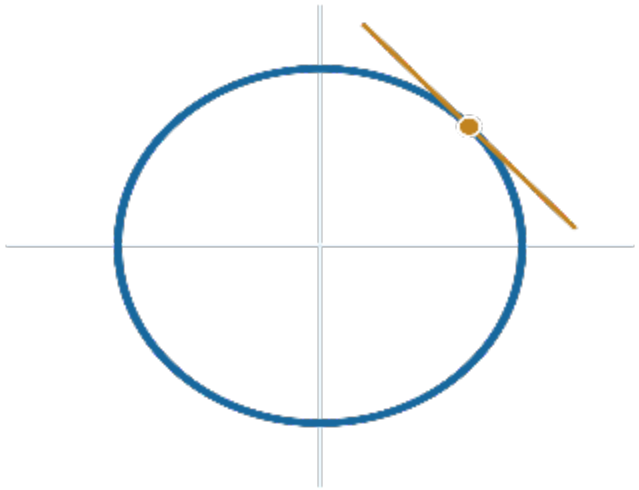
Projective closure of a hyperbola

$$x_1 x_2 = x_0^2$$

Two points at infinity

Dimension of a variety

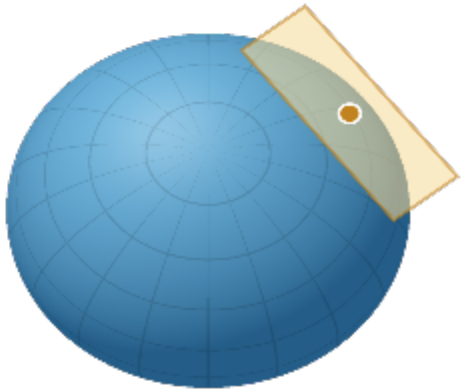
Dimension is the most important invariant of a variety.
At a general smooth point, it is the dimension of the tangent space.



Curve

$x^2 + y^2 = 1$

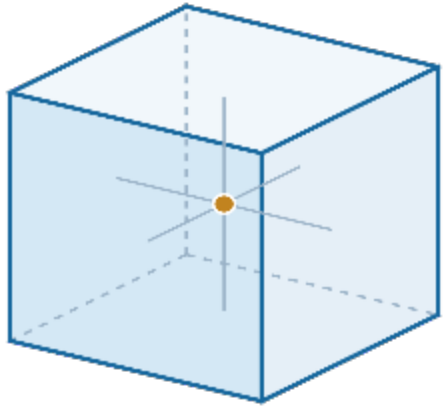
Dimension 1



Surface

$x^2 + y^2 + z^2 = 1$

Dimension 2



Threefold

$\mathbb{A}^3_{\mathbb{C}}$

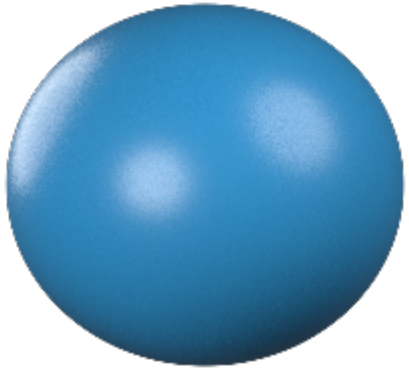
Dimension 3

Smooth projective curves

A smooth projective complex curve is a compact Riemann surface.

Its main invariant is the genus, the dimension of the first cohomology group of the structure sheaf.

$$g(C) = \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C)$$



Genus 0



Genus 1

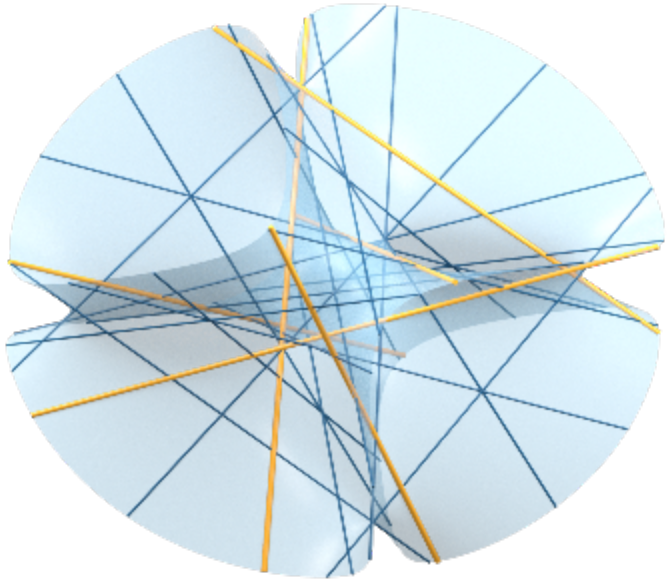


Genus 2

Smooth projective surfaces

Smooth projective surfaces are much harder to classify.

Two non-isomorphic surfaces can share a dense open subset.

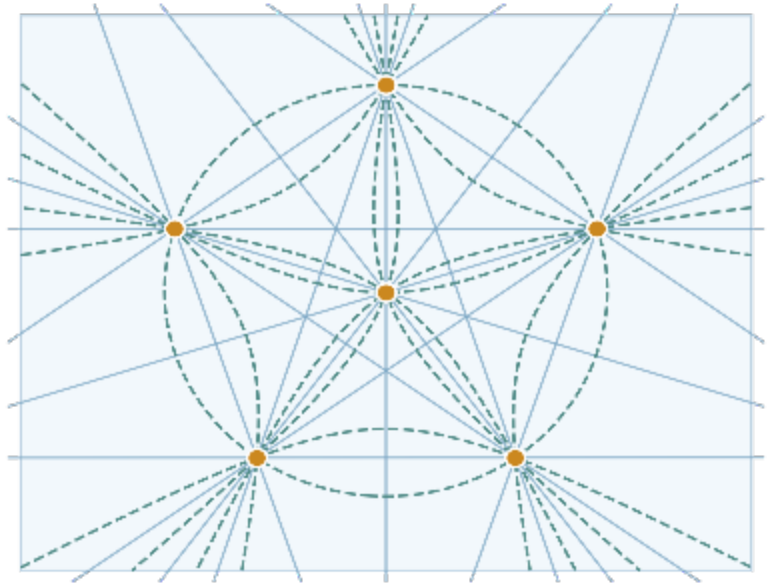


Clebsch cubic surface

$\sum_{i=0}^{4} x_i = \sum_{i=0}^{4} x_i^3 = 0 \quad \text{in } \mathbb{P}^4$

27 lines, with six disjoint lines in gold

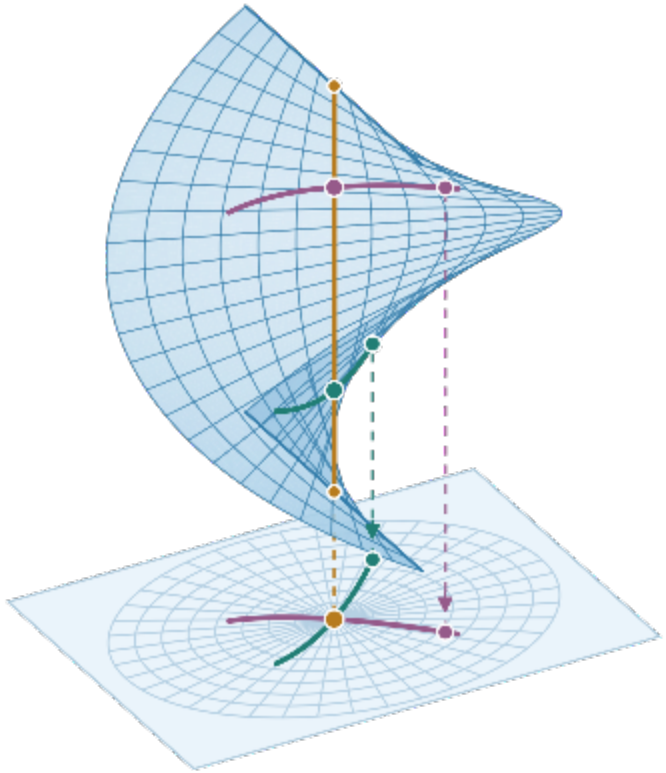
Blow down



Projective plane $\mathbb{P}^2$

Six gold points, 15 blue lines, six dashed conics

Blow-up of a surface at a point



Colored arrows match each marked point to its unique lift.

The gold exceptional curve maps to the origin.

The incidence model

$$Y = \mathrm{Bl}_0(\mathbb{A}^2) \subset \mathbb{A}^2 \times \mathbb{P}^1$$

$$Y = \big\{\big((x,y),[u:v]\big) \,:\, xv = yu\big\}$$

$\mathbb{P}^2$ version: $[u:v:w], \qquad xv = yu, \quad w = 0$

The blow-down forgets the direction

$$\pi\colon \big((x,y),[u:v]\big) \longmapsto (x,y)$$

Its inverse away from the origin

$$(x,y) \longmapsto \big((x,y),[x:y]\big)$$

$$\pi^{-1}(0) = \{0\} \times \mathbb{P}^1$$

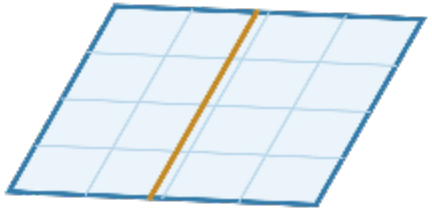
The node separates into two points upstairs.

$$y^2 = x^2(1+x)$$

Minimal surfaces

On a minimal surface, no curve can be contracted to a smooth point on another smooth projective surface.

Kodaira dimension organizes minimal surfaces by canonical positivity.



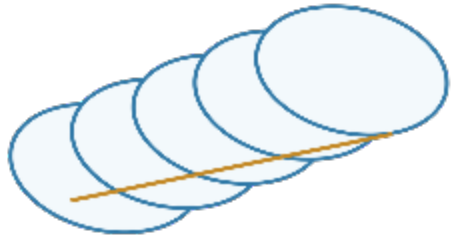
Rational and ruled

$\kappa = -\infty$



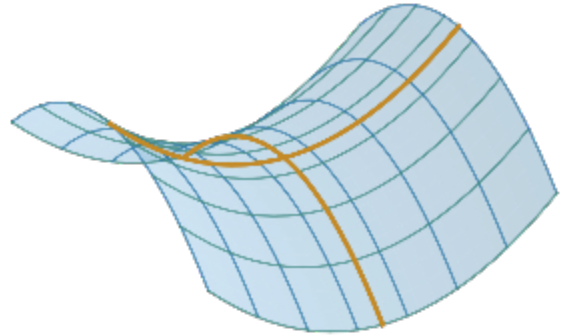
Abelian, K3 and others

$\kappa = 0$



Properly elliptic

$\kappa = 1$



General type

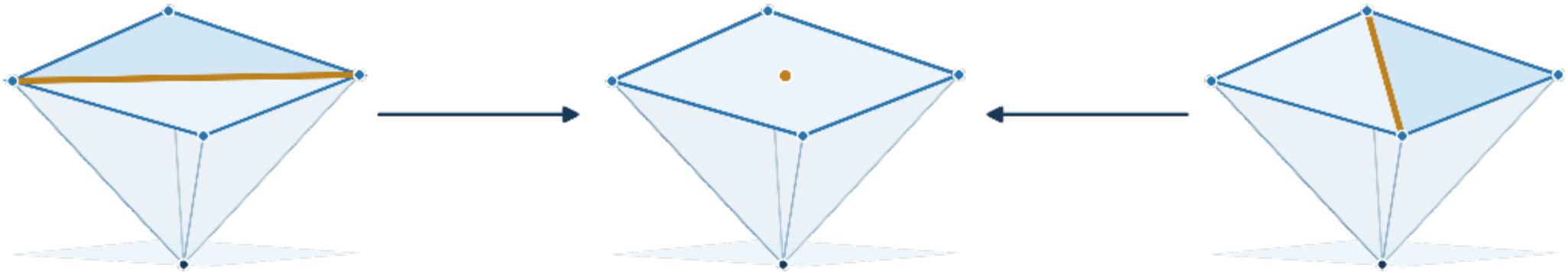
$\kappa = 2$

General type geography $\quad \mathrm{vol}(K_X) = K_X^2, \quad c_2(X)$

Flops and flips

In a threefold, a local birational surgery can replace a rational curve.

The two threefolds are isomorphic away from these curves.



Rational curve **Common contraction** **Replacement curve**

Flop $K \cdot C = 0 \quad \longrightarrow \quad K^+ \cdot C^+ = 0$

Flip $K \cdot C < 0 \quad \longrightarrow \quad K^+ \cdot C^+ > 0$

In the minimal model program, flips make ω_X more positive along curves.

So, what is birational geometry?

Birational geometry studies how to modify algebraic varieties by birational surgeries to solve geometric problems.

Singularities

Simplify the local geometry of a variety.

Ample models

Understand line bundles and vector bundles on a variety.

Enumerative geometry

Count special subvarieties.

Minimal models

Classify smooth projective varieties.

Major achievements in Birational Geometry

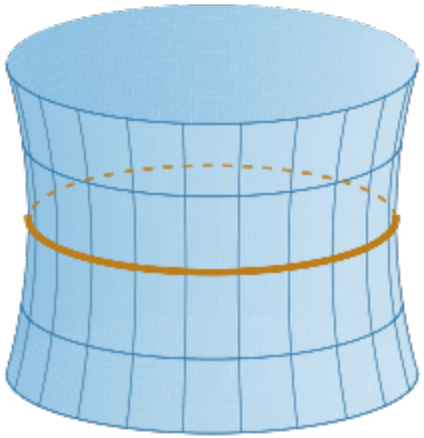
UCLA College | Physical Sciences Mathematics

Resolution of singularities

Theorem (Hironaka, 1964)

Let X be a complex algebraic variety. There exists a projective birational morphism $f : Y \longrightarrow X$ with Y smooth.

The map is an isomorphism over the smooth locus of X. If X is projective, so is Y.

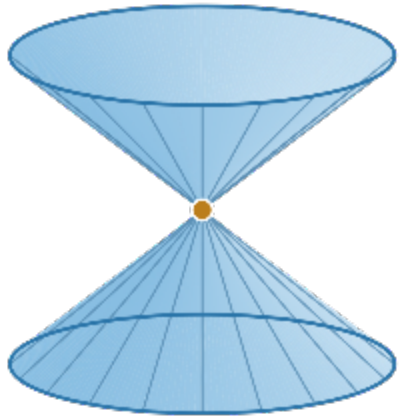


Smooth resolution

$E \simeq \mathbb{P}^1$

Contracts the exceptional curve





Quadric cone

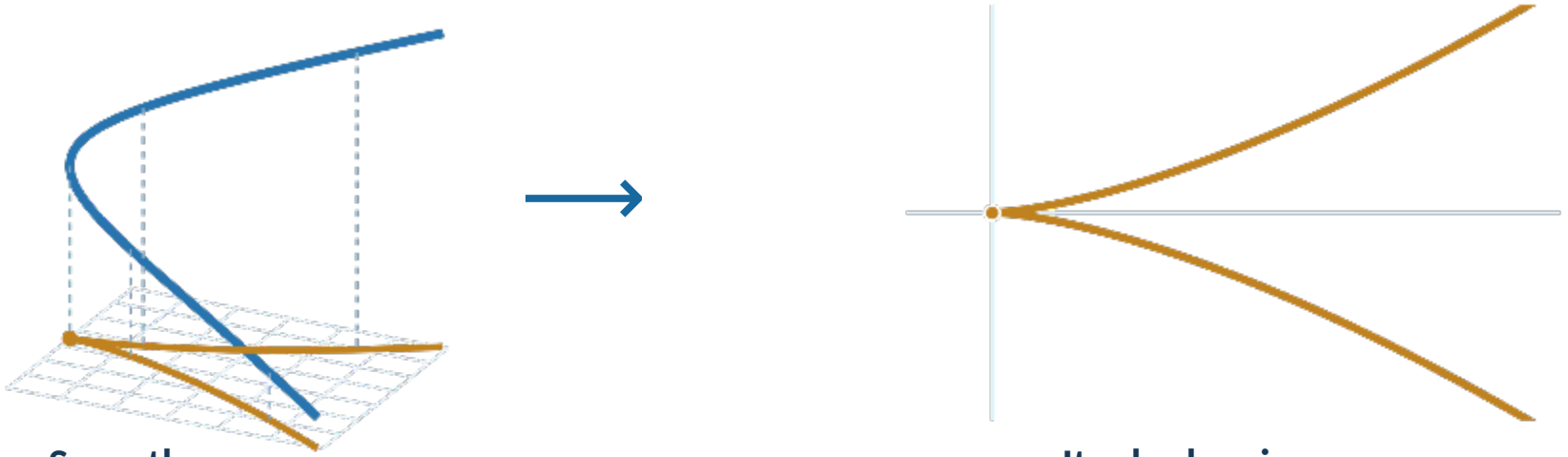
$xz = y^2$

What is a projective birational morphism?

A projective morphism can be realized as a projection.

$$Y \hookrightarrow X \times \mathbb{P}^m \longrightarrow X$$

Birational means it is an isomorphism over a dense open subset.



Smooth space curve

$(x, y, t) = (t^2, t^3, t)$

Its shadow is a cusp

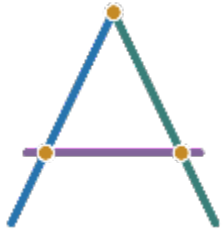
$y^2 = x^3$

"Every variety is the shadow of a smooth variety."

Resolving curve singularities

Normalization separates local branches and smooths cusps.

A



$y\big((y-1)^2 - x^2\big) = 0$

3 nodes

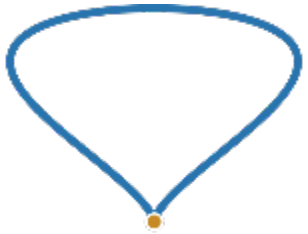
b



$x\big((x-1)^2 + y^2 - 1\big) = 0$

1 tacnode

O



$y^2 = x^3(1-x)$

1 cusp

V



$x^2 = y^3$

1 cusp

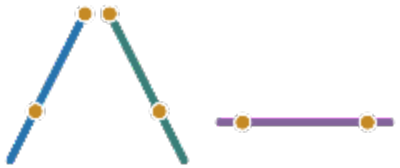
Y



$x(y^2 - x^2) = 0$

1 ordinary triple point

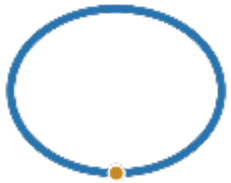
Normalizations



Three lines separate



Line and conic separate



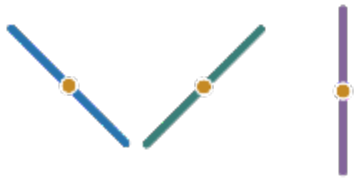
Smooth conic

$(x, w) \longmapsto (x, xw)$



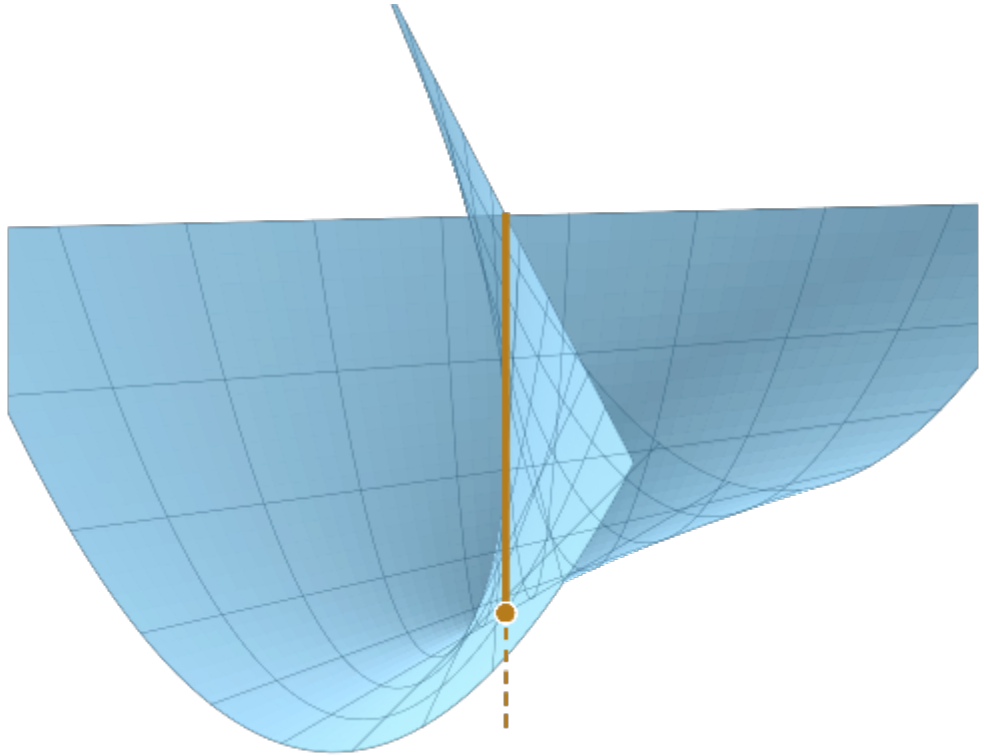
Smooth affine line

$t \longmapsto (t^3, t^2)$

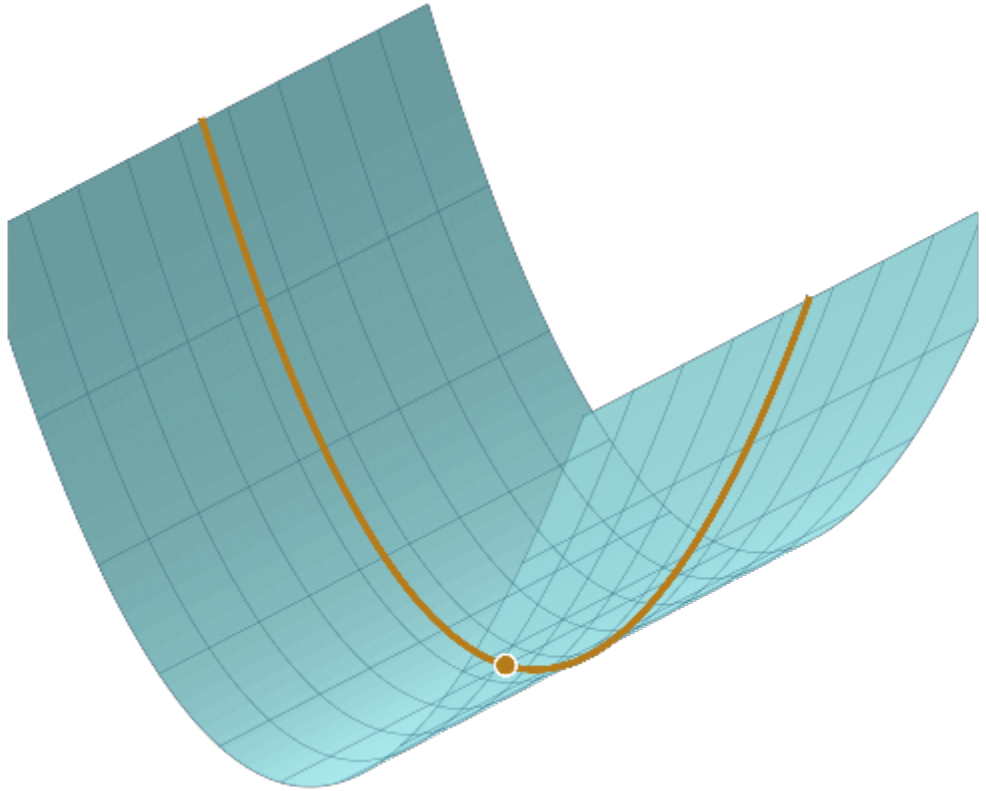


Three lines separate

Gold marks singular points above and their preimages below.



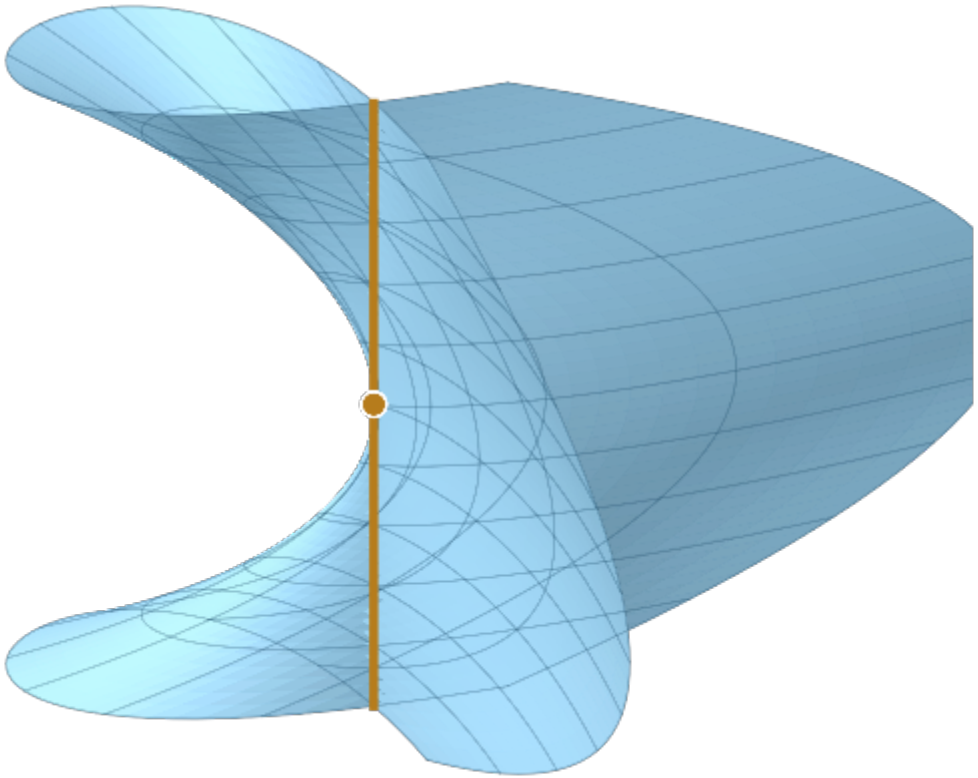
π



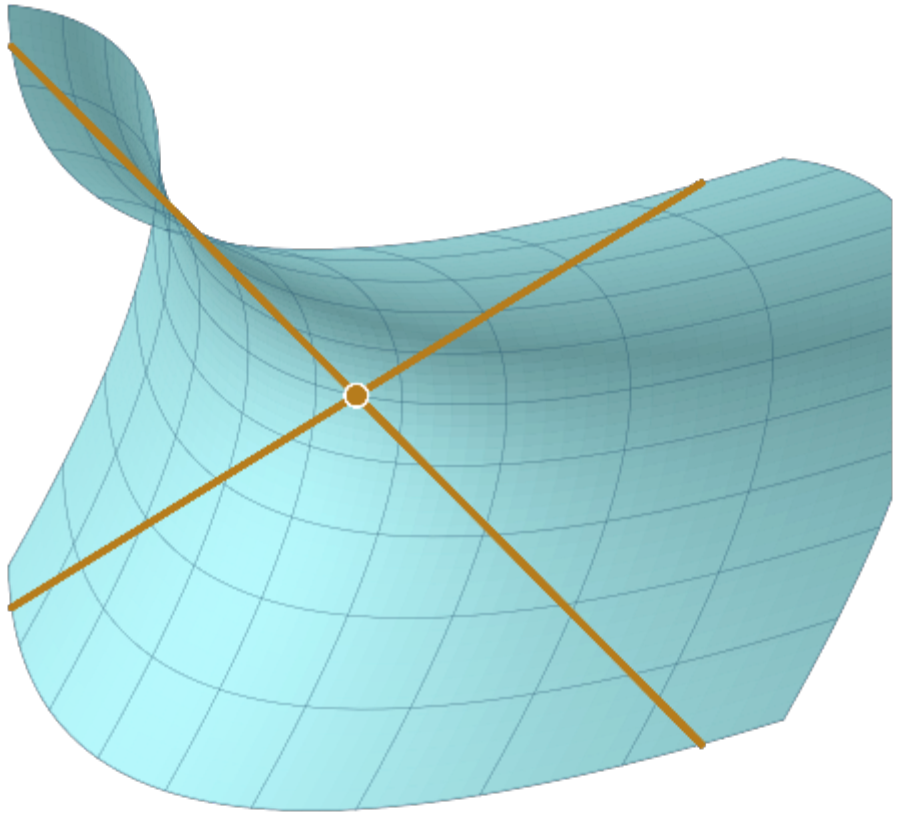
$$\underbrace{\{x^2 = y^2 z\}}_{\text{Whitney umbrella}}$$

$$\widetilde{W} = \{z = t^2\} \subset \mathbb{A}^3_{t,y,z}$$

$$\pi(t, y, z) = (ty, y, z)$$



π



$$\underbrace{\{y^2 = x^2z^2 + x^3\}}_{\text{Kolibri}}$$

$$\widetilde{K} = \{x = t^2 - z^2\} \subset \mathbb{A}^3_{t,x,z}$$

$$\pi(t, x, z) = (x, tx, z)$$

Smooth Fano varieties

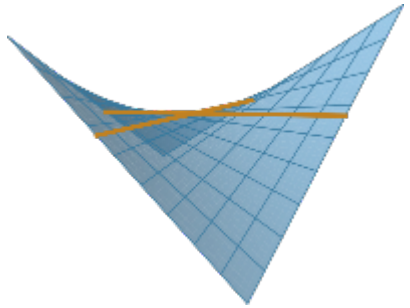
A smooth projective variety is Fano if its anticanonical divisor is ample.

Geometrically, this corresponds to positive Ricci curvature.

Theorem (Del Pezzo, 1887; modern treatment: Demazure, 1980)

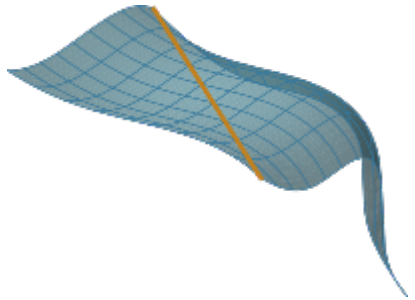
$$\mathbb{P}^1 \times \mathbb{P}^1 \quad \text{or} \quad \mathrm{Bl}_{p_1,\dots,p_r}\mathbb{P}^2, \qquad 0 \leq r \leq 8$$

Every smooth Fano surface has one of these forms. The blow-up points are in general position.



Smooth quadric surface

$$x_0x_3 = x_1x_2$$



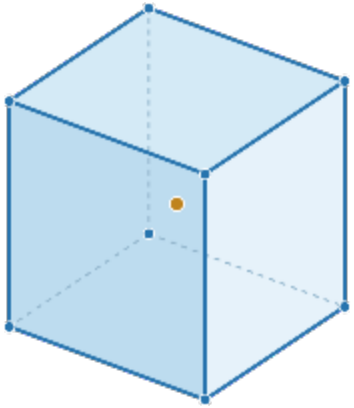
Smooth cubic surface

$$x_0^3 + x_1^3 + x_2^3 = x_3^3$$

Smooth Fano threefolds

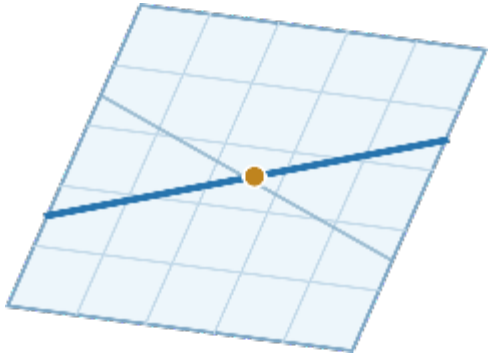
Theorem (Iskovskikh, 1977–78; Mori–Mukai, 1981–82)

Smooth complex Fano threefolds form exactly 105 deformation families.



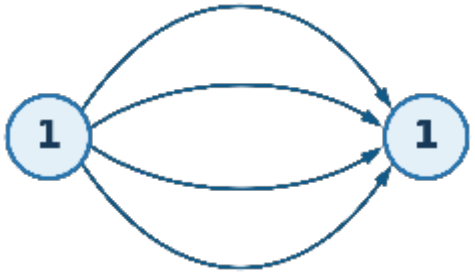
Reflexive cube

$(\mathbb{P}^1)^3$



Point–line flag

$\mathrm{Fl}(1,2;\mathbb{C}^3)$



Four-arrow Kronecker quiver

$\mathbb{P}^3$

The proof used extremal contractions and blow-ups. Mori and Mukai's 2003 erratum added one family.

The canonical ring

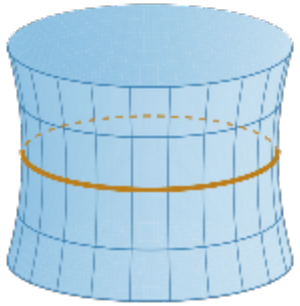
For a normal projective variety, start on its smooth locus.

$$T_U \longmapsto \Omega^1_U = T^\vee_U \longmapsto \omega_U = \bigwedge^n \Omega^1_U$$

Extend the canonical sheaf and take sections of its reflexive powers.

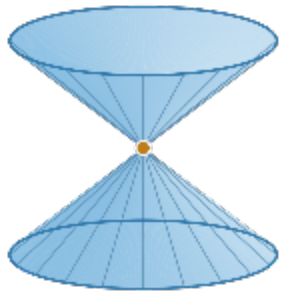
$$R(X) = \bigoplus_{m \geq 0} H^0\left(X, \omega_X^{[m]}\right)$$

For smooth projective varieties of general type, $X_{\text{can}} = \operatorname{Proj} R(X)$ is a birational model.



Smooth minimal surface

$C^2 = -2$



Canonical model

A_1 singularity

Finite generation of the canonical ring

Theorem (Birkar, Cascini, Hacon and McKernan, 2006)

The canonical ring of every smooth projective complex variety is finitely generated.

A finite set of homogeneous sections generates the entire ring.

$$\mathbb{C}[s_0, \dots, s_N] \twoheadrightarrow R(X)$$

Example (smooth plane quartic)

$$C = \{x^4 + y^4 = z^4\} \subset \mathbb{P}^2, \qquad \omega_C \simeq \mathcal{O}_C(1)$$

$$R(C) \simeq \mathbb{C}[x, y, z]/(x^4 + y^4 - z^4)$$

Three degree-one generators and one quartic relation.

Finite generation gives a projective canonical model in the general-type case.

Orbifold singularities

An orbifold is locally a smooth space modulo a finite group.

$$\mathbb{C}^n/G, \qquad G \leqslant \mathrm{GL}_n(\mathbb{C}) \text{ finite}$$

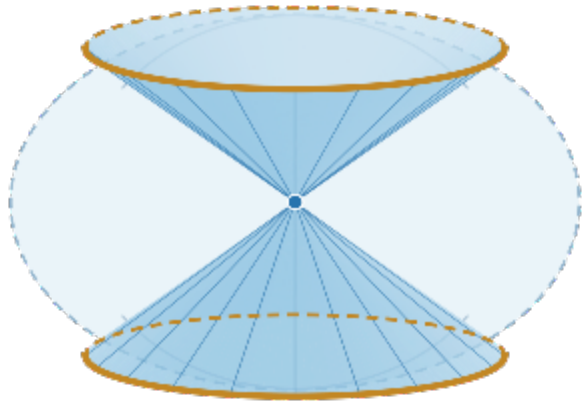


Sign action on a smooth space

$$(z, w) \sim (-z, -w)$$

Quotient cone

$$(a, b, c) = (z^2, zw, w^2), \qquad ac = b^2$$



Link of the singularity

$$L_{A_1} \cong S^3/\{\pm 1\} \cong \mathbb{RP}^3$$

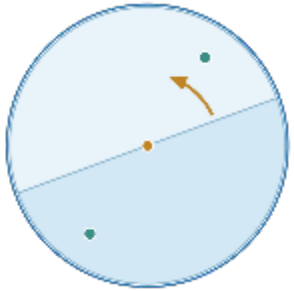
The link is the intersection with a small sphere around the singular point.

The local canonical index

At a point of an orbifold, the index is the smallest positive integer r such that

$$rK_X \text{ is Cartier near } p$$

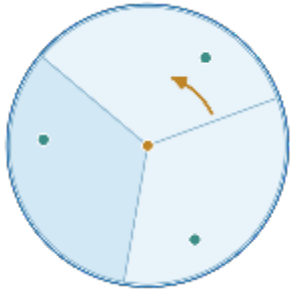
It measures which power of a local canonical form descends.



Index 1

$$(z, w) \mapsto (-z, -w)$$

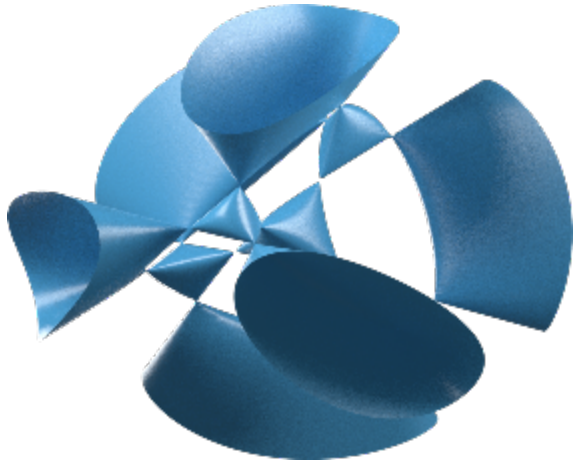
$$dz \wedge dw \mapsto dz \wedge dw$$



Index 3

$$(z, w) \mapsto (\zeta z, \zeta w), \quad \zeta = e^{2\pi i/3}$$

$$dz \wedge dw \mapsto \zeta^2\, dz \wedge dw$$



Kummer quartic

$$A/\{\pm 1\}: \quad 16 \text{ nodes of index } 1$$

Boundedness of Fano varieties

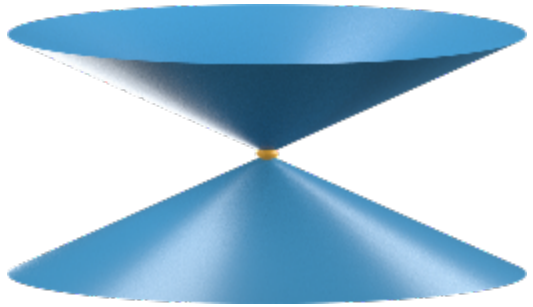
Theorem (Birkar, 2021)

Fix a dimension and a bound on the local canonical index.

Fano varieties with orbifold singularities of bounded local index form a bounded family.

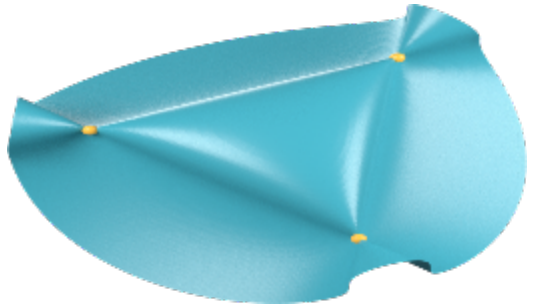
Every such Fano variety appears as a fiber of one family over a finite-type base.

Rank-one Gorenstein del Pezzo surfaces (Miyanishi–Zhang, 1988)



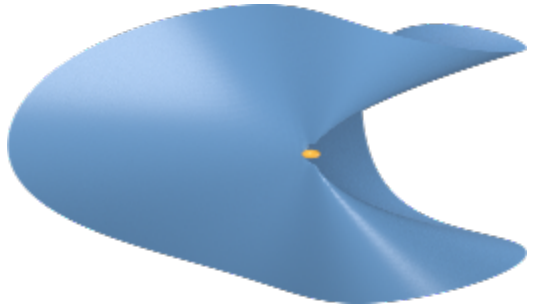
A_1 singularity

$x^2 + y^2 = z^2$



$3A_2$ singularities

$xyz = w^3$



E_6 singularity

$x^2w + xz^2 + y^3 = 0$

The impact of AI in Birational Geometry

OpenAI announcements

Three results announced in OpenAI's 2026 AI-generated preprints

A solution to the abundance conjecture

For smooth projective varieties, $\omega_X \text{ nef} \implies \omega_X \text{ semiample}$

Termination of fourfold flips

For projective log canonical fourfolds with rational boundary, the minimal model program stops.

The flips used to improve canonical positivity cannot go on indefinitely.

Existence of klt complements

Bounded klt complements exist locally for $\varepsilon\text{-log canonical}$ Fano contractions.

These complements control the local behavior of klt singularities.

The proof seems locally correct, and probably they are globally correct. However, they don't yet add a lot to our human understanding.

Opportunities for AI + human insight

Classification of varieties

Generate and compare candidates for smooth Fano fourfolds and singular Fano threefolds.

New birational invariants

Use Hodge theory to look for quantities that survive birational modifications.

Varieties with extremal behavior

Search for varieties with small volume that could make volume bounds sharp.

Major field restructuring

Which definitions are necessary? Could revising them lead to cleaner proofs?

How do I use AI?

- I use AI as a secondary source of ideas. Once I have some good ideas to tackle a research problem, I play with AI to combine them and possibly create secondary ideas.
- Create databases of algebraic varieties to test conjectures.
- I try to be extremely careful with properly citing human ideas and not scooping other people.
- I also use AI heavily to check for correctness.

Summary

“Birational geometry is a field of mathematics that is about 125 years old. Mathematicians guided its development and made several major breakthroughs before 2025. Since 2025, AI has helped generate new ideas and advance the field. Yet much human insight is still missing, even in completed papers written by AI. We need more researchers (undergraduates, graduate students, postdocs, and faculty) to keep advancing the field and uncover the insights we’re still missing.”

Thank you for your attention